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# Exact solution of the convex polygon perimeter and area generating function

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**Abstract.** An explicit expression is derived for the three-variable generating function $P(x, y, z) = \Sigma_{m\geq 1} \Sigma_{n\geq 1} \Sigma_{r\geq 1} x^{2n}y^{2m}z^r c_{n,m,r}$, where $c_{n,m,r}$ is the number of convex polygons with horizontal width $n$, vertical height $m$ and area $r$.

## 1. Introduction

The self-avoiding polygon (SAP) was considered as a model of crystal growth (Temperley 1952) or polymer (Temperley 1956, de Gennes 1979, Privman and Svrakic 1988). The problem in two dimensions is to find the generating function for the number of polygons on a lattice with definite perimeter and/or area. An exact solution has not yet been found. However, simpler SAP problems can be solved. In particular there are four classes of SAPs on the square lattice where exact solutions were obtained; these are the pyramid polygons, staircase polygons, convex and row-convex polygons.

A convex polygon on the square lattice has the property that a straight line on the bonds of the dual lattice cuts the bonds of the polygon at most twice. The pyramid-like polygon is a special case of the convex polygon such that the width at the bottom equals the width of the bounding rectangle. Temperley (1952) suggested long ago that the pyramid-like polygon can be considered as a two-dimensional model of the growth of a crystal on a plane substrate. He obtained the generating function for the number of such polygons with fixed area. Recently Lin (1991) generalized the result of Temperley and derived the generating function for the number of polygons with fixed values of height, width and area.

The three-variable generating function for polygons on the square lattice is defined by

$$P(x, y, z) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \sum_{r=1}^{\infty} x^{2n}y^{2m}z^r c_{n,m,r} \tag{1}$$

where $c_{n,m,r}$ is the number of polygons with $2n$ horizontal steps, $2m$ vertical steps and area $r$. Exact solutions of (1) have been found for the staircase polygons (Temperley 1956, Pólya 1969, Lin *et al* 1987, Brak and Guttmann 1990, Lin and Tzeng 1991) and row-convex polygons (Temperley 1956, Brak *et al* 1990, Brak and Guttmann 1990, Lin 1990b, Lin and Tzeng 1991).

The perimeter generating function, which is a special case of (1) with $x = y$ and $z = 1$, for convex polygons was first obtained by Delest and Viennot (1984) and then rederived by simpler methods (Kim 1988, Guttmann and Enting 1988, Lin and Chang

1988). The perimeter and area generating function for convex polygons has not been solved. The $r$th area-weighted moment is related to the $r$th partial derivative of (1) with respect to $z$ evaluated at the point $z=1$. A general method to obtain the $r$th moment was given by Lin (1990a). Lin used the computer algebra program REDUCE (Hearn 1968, Stauffer *et al* 1989) to calculate the first ten moments, and his result agrees with the non-rigorous result of Enting and Guttmann (1989) for the first two moments.

In this paper we derive the exact three-variable generating function for convex polygons in section 4. We review the pyramid polygon and staircase polygon generating functions respectively in sections 2 and 3.

## 2. Pyramid polygon

Consider a pyramid polygon on the square lattice as shown in figure 1. The generating function can be written in the form

$$G(x, y, z)=\sum_{m=1}^{\infty} g_m(x, y, z) \tag{2}$$

where $g_m$ is the generating function for all pyramid polygons whose width at the bottom is $m$. Using the method of privman and Svrakic (1988), Lin (1991) proved that

$$\sum_{m=1}^{\infty} t^{m-1} g_m(x, y, z)=R(x, y, z, t)=x^2 \sum_{n=1}^{\infty} (y^2 z)^n (1-tx^2 z^n) Q_n^{-2} \tag{3}$$

where

$$Q_n=\prod_{k=1}^{n} (1-tx^2 z^k). \tag{4}$$

Consequently the generating function is

$$G(x, y, z)=R(x, y, z, 1)=x^2 \sum_{n=1}^{\infty} \left[\frac{(y^2 z)^n (1-x^2 z^n)}{\prod_{k=1}^{n} (1-x^2 z^k)^2}\right]. \tag{5}$$

The special case of $x=y=1$ was first derived by Temperley (1952). Another special case of $z=1$ was first obtained by Lin and Chang (1988).

Using the identity (Andrews 1976)

$$\left(\prod_{m=0}^{n} (1-tz^m)\right)^{-1}=\sum_{k=0}^{\infty} t^k q_{n+k}(z)[q_n(z) q_k(z)]^{-1} \tag{6}$$

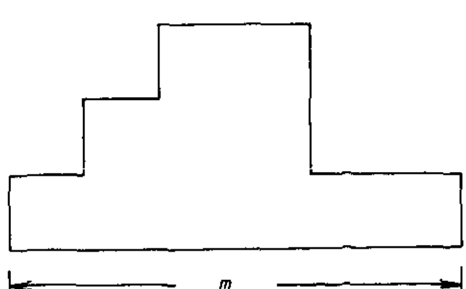


Figure 1. A pyramid polygon on the square lattice with width $m$.

where

$$q_n = \begin{cases} \prod_{k=1}^{n} (1-z^k) & n>0 \\ 1 & n=0 \end{cases}$$

it can be shown that

$$Q_n^{-2} = q_n^{-2}(1-x^2 t)^2 \sum_{k=0}^{\infty} (x^2 t)^k t_{k,n} \tag{7}$$

$$g_m = x^{2m} \sum_{n=1}^{\infty} (y^2 z)^n q_n^{-2}[t_{m-1,n} - (2+z^n)t_{m-2,n} + (1+2z^n)t_{m-3,n} - z^n t_{m-4,n}] \tag{8}$$

where

$$t_{k,n} = \begin{cases} \sum_{r=0}^{k} q_{n+r}(z) q_{n+k-r}(z)/q_r(z) q_{k-r}(z) & k \geq 0 \\ 0 & k<0. \end{cases} \tag{9}$$

## 3. Staircase polygon

Consider a staircase polygon on the square lattice as shown in figure 2. The generating function is

$$H(x, y, z) = \sum_{m=0}^{\infty} h_m(x, y, z) \tag{10}$$

where $h_m$ is the generating function for all staircase polygons whose top row contains $m$ squares. The special case of $x=y$ was solved by Brak and Guttmann (1990) and the general case by Lin and Tzeng (1991). The result is

$$h_n = y^2(x^2 z)^n \left(1 + \sum_{m=1}^{\infty} R_m z^{m(m+1+2n)/2}\right)\left(1 + \sum_{m=1}^{\infty} R_m z^{m(m+1)/2}\right)^{-1} \tag{11}$$

$$H(x, y, z) = x^2\left(1 + \sum_{m=1}^{\infty} R_m z^{m(m+3)/2}\right)\left(1 + \sum_{m=1}^{\infty} R_m z^{m(m+1)/2}\right)^{-1} - x^2 \tag{12}$$

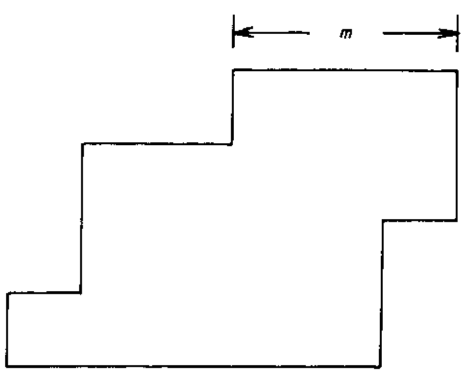


**Figure 2.** A staircase polygon.

where

$$R_m = \frac{(-y^2)^m}{\prod_{r=1}^{m} (1-z^r)(1-x^2z^r)}. \tag{13}$$

## 4. Convex polygon

It was shown by Lin and Chang (1988) that a convex polygon can be divided by a horizontal line into two (top and bottom) polygons such that the top polygon is a pyramid polygon with maximum possible area. The generating function for convex polygons is given by (Lin and Chang 1988, Lin 1990a)

$$P(x, y, z) = G + 2 \sum_{m=2}^{\infty} g_m \sum_{n=1}^{m-1} x^{-2n} \sum_{p=0}^{\infty} f_{n+p} + \sum_{m=3}^{\infty} g_m S_m \tag{14}$$

where

$$S_m = \sum_{n=1}^{m-2} g_n x^{-2n} (m-n-1) \tag{15}$$

$G$ and $g_m$ are defined by (5) and (8) respectively. In (14), $f_m$ is the generating function for a special class of convex polygons (see figure 3) whose top row contains $m$ squares. The top right-hand corner of such a polygon is also a corner of the bounding rectangle. Notice that such a polygon can be divided by a horizontal line uniquely into two polygons such that the top polygon is a staircase polygon with maximum possible area and the bottom polygon is an inverse pyramid polygon as shown in figure 3. Consequently we have

$$f_m = h_m + \sum_{n=2}^{\infty} h_{m,n} S_{n+1} \tag{16}$$

where $h_{m,n}$ is the generating function for staircase polygons whose top and bottom rows contain respectively $m$ and $n$ squares. The first term in the RHS of (16) corresponds to the special case where the bottom polygon does not exist.

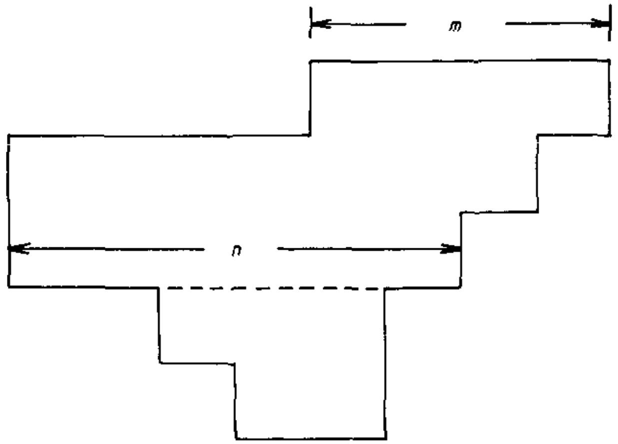


**Figure 3.** A convex polygon whose top right-hand corner is a corner of the bounding rectangle.

The generating function $h_{m,n}$ is given by

$$h_{m,n}=h_{n,m}=A[h_m(h'_n-h_n)]+\delta_{m,n}y^2z^nx^{2n} \qquad m\geqslant n \tag{17}$$

where

$$\begin{aligned} h'_n&=y^2z^n\left(1+\sum_{m=1}^{\infty}R'_m z^{m(m+1+2n)/2}\right)\left(1+\sum_{m=1}^{\infty}R'_m z^{m(m+1)/2}\right)^{-1}\\ R'_m&=\frac{(-y^2)^m}{\Pi_{r=1}^{m}(1-z^r)(x^2-z^r)}\\ A&=y^2z/(h'_1-h_1). \end{aligned} \tag{18}$$

The derivation of (17) is given in the appendix.

Substituting (16) and (17) into (14), we finally obtain the generating function for the convex polygons on the square lattice:

$$\begin{aligned} P(x,y,z)=G&+\sum_{m=3}^{\infty}g_mS_m+2y^2\sum_{m=2}^{\infty}g_m\sum_{n=1}^{m-1}x^{-2n}\sum_{r=n}^{\infty}z^rx^{2r}S_{r+1}\\ &+2\left(\sum_{m=2}^{\infty}g_mT_m\right)\left(1+A\sum_{r=2}^{\infty}S_{r+1}(h'_r-h_r)\right)\\ &+2A\sum_{m=2}^{\infty}g_m\sum_{n=1}^{m-1}x^{-2n}\sum_{r=n+1}^{\infty}S_{r+1}\sum_{p=0}^{r-n-1}(h_rh'_{n+p}-h'_rh_{n+p}) \end{aligned} \tag{19}$$

where

$$\begin{aligned} T_m&=\sum_{n=1}^{m-1}x^{-2n}\sum_{p=0}^{\infty}h_{n+p}\\ &=y^2\left(\sum_{k=0}^{\infty}R_kz^{k(k+1)/2}(z^{k+1}-z^{m(k+1)})[(1-x^2z^{k+1})(1-z^{k+1})]^{-1}\right)\\ &\quad\times\left(\sum_{k=0}^{\infty}R_kz^{k(k+1)/2}\right)^{-1} \end{aligned} \tag{20}$$

$R_0=1$ and $S_2=0$. We have expanded (19) to 20th order in $x$ and $y$, and the result agrees with the exact counting of the convex polygons.

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## Appendix

The generating function $h_{m,n}$ satisfies the following relation

$$h_{m,n}=y^2z^n\left(x^{2n}\delta_{m,n}+\sum_{r=1}^{n}x^{2(n-r)}\sum_{k=0}^{\infty}h_{m,r+k}\right). \tag{A1}$$

Equation (A1) is derived as follows. The first term on the RHS represents a one-row polygon which contains $n$ squares. The second term represents the different ways to put a staircase polygon on the top of this one-row polygon as shown in figure 4. Using the identity

$$h_m = \sum_{k=1}^{\infty} h_{m,k} \tag{A2}$$

we have

$$\begin{aligned} h_{m,1} &= y^2 z(x^2 \delta_{m,1} + h_m) \\ h_{m,2} &= y^2 z^2[x^4 \delta_{m,2} - x^2 y^2 z \delta_{m,1} + h_m(1 + x^2 - y^2 z)]. \end{aligned} \tag{A3}$$

It follows from (A1) that we have the recursion relation

$$\begin{aligned} h_{m,n+2} - z(1 + x^2 - y^2 z^{n+1}) h_{m,n+1} + x^2 z^2 h_{m,n} \\ = y^2 z^{n+2} x^{2(n+1)}[x^2 \delta_{m,n+2} - (1 + x^2)\delta_{m,n+1} + \delta_{m,n}]. \end{aligned} \tag{A4}$$

The boundary condition (A3) implies that the solution of equation (A4) can be written in the form

$$h_{m,n} = \sum_{r=1}^{n} \delta_{m,r} d_{r,n} + h_m p_n. \tag{A5}$$

Substituting (A5) into (A4), we get

$$p_{n+2} - z(1 + x^2 - y^2 z^{n+1}) p_{n+1} + x^2 z^2 p_n = 0 \qquad n = 1, 2, \ldots \tag{A6}$$

$$d_{r,n+2} - z(1 + x^2 - y^2 z^{n+1}) d_{r,n+1} + x^2 z^2 d_{r,n} = 0 \qquad r < n \tag{A7}$$

$$d_{n,n} = y^2 z^n x^{2n} \tag{A8}$$

$$d_{n-1,n} = -y^4 z^{2n-1} x^{2(n-1)} \tag{A9}$$

$$d_{n-2,n} = -y^4 z^{2n-2} x^{2(n-2)}(1 + x^2 - y^2 z^{n-1}). \tag{A10}$$

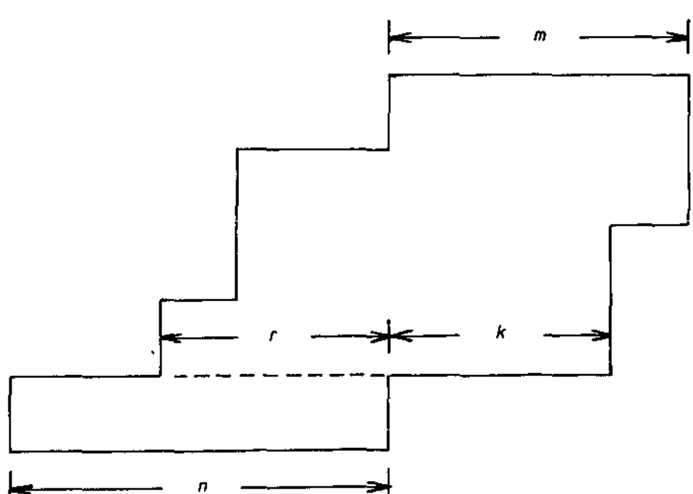


**Figure 4.** A staircase polygon can be divided into a top (staircase) polygon and a bottom (one-row) polygon by a horizontal line.

The recursion relation (A6) has been solved by Lin and Tzeng (1991) and the solution is†

$$p_n = Ah'_n + A'h_n \tag{A11}$$

where $h_n$ and $h'_n$ are defined by (11) and (18) respectively. The coefficients $A$ and $A'$ are determined by the boundary condition

$$p_1 = y^2 z \qquad p_2 = y^2 z^2(1 + x^2 - y^2 z). \tag{A12}$$

The result is

$$A = -A' = y^2 z/(h'_1 - h_1). \tag{A13}$$

Similarly the solution of (A7) is

$$d_{r,n} = B_r h'_n + B'_r h_n \qquad r < n \tag{A14}$$

where the coefficients $B_r$ and $B'_r$ are determined by the boundary condition

$$d_{r,r+1} = B_r h'_{r+1} + B'_r h_{r+1} \qquad d_{r,r+2} = B_r h'_{r+2} + B'_r h_{r+2}. \tag{A15}$$

The result is

$$B_r = -Ah_r \qquad B'_r = Ah'_r. \tag{A16}$$

Substituting (A8), (A11), (A13), (A14), and (A16) into (A5), we get

$$h_{m,n} = A\left( h_m(h'_n - h_n) + \sum_{r=1}^{n-1} \delta_{m,r}(h'_r h_n - h_r h'_n) \right) + \delta_{m,n} d_{n,n}. \tag{A17}$$

It is simple to show that $h_{m,n} = h_{n,m}$ as it should be from the symmetry of the staircase polygon. It follows from (A17) that

$$h_{m,n} = h_{n,m} = A[h_m(h'_n - h_n)] + \delta_{m,n} y^2 z^n x^{2n} \qquad m \geqslant n. \tag{A18}$$

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† Our equation (A6) differs from theirs by the exchange of $x$ and $y$.